# Transition Maths and Algebra with Geometry

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#### Lecture Notes Electrical and Computer Engineering









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## System of linear equations

#### Definition

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots a_{1n} \cdot x_n = b_1,$$
  
 $a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots a_{2n} \cdot x_n = b_2,$   
 $\dots$ 

$$a_{m1}\cdot x_1+a_{m2}\cdot x_2+\ldots a_{mn}\cdot x_n=b_m.$$

is a system of *m* linear equations with *n* uknowns:  $x_1, \ldots x_n$ .

#### Definition

A solution of the above system is a set of values for the unknowns  $x_1 = k_1, \ldots, x_n = k_n$  which satisfy the above *m* equalities.

# Example of SoLE

$$3x + y + z = 1,$$
$$x + y = 0.$$

This is a system of 2 equations with 3 unknowns. It is easy to check that x = 0, y = 0, z = 1 or x = -1, y = 1, z = 3 is a solution (there are other!).

$$\begin{aligned} x + y &= 1, \\ x + y &= 0. \end{aligned}$$

This is a system of 2 equations with 2 unknowns. It has no solutions since if there were any then it would mean that 0 = 1.



## Matrix representation

#### Observation

A system of *m* linear equations with  $x_1, \ldots, x_n$  as unknowns:

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n = b_1,$$
  

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n = b_2,$$
  

$$\dots$$
  

$$a_{m1} \cdot x_1 + a_{k2} \cdot x_2 + \ldots + a_{mn} \cdot x_n = b_m.$$

can be represented by a matrix equation:

 $A \cdot X = B$ 

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

# Equivalence

#### Definition

Two systems  $A_1X = B_1$  and  $A_2X = B_2$  of *m* linear equations with *n* unknowns are said to be *equivalent* if they have the same solution set.

Example:

$$\begin{aligned} x - y &= 1, \\ x + y &= 0. \end{aligned}$$

is equivalent to

$$2x - 2y = 2,$$
$$x + y = 0.$$



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## Augmented matrix

#### Definition

The matrix

$$(A|B) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

is called an augmented matrix of the system  $A \cdot X = B$ .

Example: the matrix

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$$\left(\begin{array}{cc|c} 2 & -2 & 2\\ 1 & 1 & 0 \end{array}\right)$$

is the augmented matrix of the system

$$2x - 2y = 2$$



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### Equivalence of systems: properties

Given two systems  $A_1X = B_1$  and  $A_2X = B_2$  when are they equivalent? How

can we show that they have the same solution set?

#### Theorem

Two systems  $A_1X = B_1$  and  $A_2X = B_2$  of *m* linear equations with *n* unknowns are equivalent iff their augmented matrices  $(A_1|B_1)$  and  $(A_2|B_2)$  are row equivalent.

Recall that two matrices are row equivalent if one can be obtained from the other by elementary row operations:

- **(Row switching)** *i*-th row and *j*-th row are interchanged  $(R_i \leftrightarrow R_j)$ ,
- **(Row scaling)** each element in *i*-th row is multiplied by a *nonzero* scalar  $k \in \mathbb{K} (kR_i \rightarrow R_i)$ ,
- **(Row addition)** *i*-th row is replaced by a sum of *i*-th row and a multiple of *j*-th row  $(R_i + k \cdot R_i \rightarrow R_i)$ .

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## SoLEs in echelon form

#### Definition

A system AX = B of linear equations is said to be in *echelon form* if the augmented matrix (A|B) is in row echelon form.

Recall that a matrix is in row echelon form if

- all zero rows are at the bottom,
- the first nonzero number in the *i*-th row is to the right from the first nonzero coefficient in the row above it.









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# Solving SoLEs

Recall that each matrix is row equivalent to a matrix in row echelon form. Therefore, for any system AX = B there exists an equivalent system A'X = B' in echelon form. In other words, there is a system A'X = B' in echelon form with the same solution set.

#### Solving SoLEs

- Given a system AX = B write its augmented matrix (A|B),
- Find a matrix (A'|B') in row echelon form which is row equivalent to (A|B),
- Write the new system A'X = B' and deduce its solutions.



## Solving SoLEs: Examples

Consider the following SoLE:

$$2x + 4y - z = 11,$$
  
-4x - 3y + 3z = 20,  
2x + 4y + 2z = 2.

Its augmented matrix is given by:



## Solving SoLEs: Examples

Using elementary row operations we obtain the following matrix in row echelon form equivalent to the matrix from the previous slide:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 11 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 3 & -9 \end{array}\right)$$









# Solving SoLEs: Examples

$$\left(\begin{array}{ccccc} 2 & 4 & -1 & | & 11 \\ 0 & 5 & 1 & | & 2 \\ 0 & 0 & 3 & | & -9 \end{array}\right)$$

This is the augmented matrix of the system:

$$2x + 4y - z = 11,$$
  

$$5y + z = 2,$$
  

$$3z = -9.$$

We see that the 3rd equation implies that z = -3. This and the 2nd equations gives us 5y - 3 = 2 and hence y = 1. Similarily we show that x = 2. The system has a unique solution x = 2, y = 1, z = -3. Back to new presentation PROGRAM ROZWOJOWY POLITECHNIKI WARSZAWSKIEJ

# Solving SoLEs: Examples

Consider the following SoLE:

$$x + 4y - 3z + 2t = 5,$$
  
 $2x + 8y - 5z = 12.$ 

It augmented matrix is:

By row reduction we get:







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## Solving SoLEs: Examples

$$x + 4y - 3z + 2t = 5,$$
$$z - 4t = 2.$$

This system in echelon form doesn't have a triangular form as in the previous example. We see that there are 2 equations and 4 unknowns. The system has more than one solution. We take non-leading variables, namely y and t, and assign arbitrary values to these. Say y = a and t = b. Then we use a simple substitution to get: z - 4b = 2 hence z = 2 + 4b and x = 11 - 4a + 10b. The solution depends on two parameters, a and b and is given by:

$$x = 11 - 4a + 10b$$
,  $y = a$ ,  $z = 2 + 4b$ ,  $t = b$ 



# Solving SoLEs: Examples

Consider the following SoLE:

$$2x + 4y = 0,$$
$$2x + 4y = 1.$$

Its augmented matrix is given by:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 1 \end{array}\right)$$

Using elementary row operations we obtain the following matrix in row echelon form equivalent to the matrix above:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 1 \end{array}\right)$$





## Kronecker-Capelli Theorem

#### Theorem

A system AX = B of linear equations has a solution iff r(A) = r(A|B).

Example:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 1 \end{array}\right)$$







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### 3 Nonhomogeneous and associated homogeneous systems









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## Homogeneous SoLEs

#### Definition

A system of linear equations is called  $\ensuremath{\textit{homogeneous}}$  if it is of the form

$$A \cdot X = 0$$

The homogeneous system of linear equations ALWAYS has a solution, namely  $x_1 = 0, x_2 = 0, ..., x_n = 0$ . Any other solution (if there is one) is called a *non-trivial* solution.



Homogeneous SoLEs:Examples

Consider the following system:

$$x + 2y - 3z + w = 0$$
$$x - 3y + z - 2w = 0$$
$$2x + y - 3z + 5w = 0$$

We see that there are 3 equations and 4 unknowns. Therefore, there are non-trivial solutions!



## Homogeneous SoLEs:Examples

Consider the following HSoLE:

$$x + y - z = 0$$
$$2x - 3y + z = 0$$
$$x - 4y + 2z = 0.$$

It can be reduced to

$$\begin{aligned} x + y - z &= 0\\ -5y + 3z &= 0. \end{aligned}$$

This system also has a non-trivial solution since if we take the non-leading variable z and assign to it any value, say z = a, then  $y = \frac{3}{5}a$  and  $x = \frac{2}{5}a$ .



## Homogeneous SoLEs:Examples

Consider the following HSoLE:

$$x + y - z = 0$$
$$2x + 4y - z = 0$$
$$3x + 2y + 2z = 0.$$

It can be reduced to

$$x + y - z = 0,$$
  

$$2y + z = 0,$$
  

$$11z = 0.$$

#### This system also has only the zero solution.



# Solution space

#### Theorem

Let AX = 0 be a system of *m* linear homogeneous equations with *n* unknowns. Then the set  $W = \{ \mathbf{v} \in \mathbb{K}^n \mid A\mathbf{v} = 0 \}$  of solutions is a subspace of the vector space  $\mathbb{K}^n$ . Moreover,

$$\dim(W) = n - r(A).$$

Proof (of the 1st part of the statement): Take  $\mathbf{v}_1, \mathbf{v}_2 \in W$ . This means that  $A\mathbf{v}_1 = 0$  and  $A\mathbf{v}_2 = 0$ . Hence,

$$A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = 0 + 0 = 0.$$

Therefore,  $\mathbf{v_1} + \mathbf{v_2} \in W$ . Similarly, we prove that for any  $k \in \mathbf{K}$ ,  $k \cdot \mathbf{v_1} \in W$ .



### Solution space and basis: Example

Consider the following HSoLE:

$$x + 2y + 3z = 0,$$
  
 $-2x - 4y - 6z = 0.$ 

It reduces to:

$$x + 2y + 3z = 0.$$

The non-leading variables are y and z. Put y = a and z = b. The solutions are of the form:



### Solution space and basis: Example

The solution space is given by:

$$W = \{ \left( egin{array}{c} -2a-3b\ a\ b \end{array} 
ight) \mid a,b\in \mathbb{R} \}.$$

We see that any vector from W depends on two parameters, namely a and b. If we put a = 1, b = 0 and a = 0, b = 1 we will obtain two vectors



which form a basis of *W*. **PROGRAM ROZWOJOWY** POLITECHNIKI WARSZAWSKIEJ







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### Homogeneous and nonhomogeneous systems

#### Theorem

Let AX = B be an arbitrary system of linear equations. Let U be the solution set and let  $\mathbf{v}_0 \in U$  be a particular solution to this system of linear equations. Then

$$U = \mathbf{v}_0 + W = \{\mathbf{v}_0 + \mathbf{w} \mid \mathbf{w} \in W\},\$$

where W is the solution space of the homogeneous system AX = 0.

Proof: any element from  $\mathbf{v}_0 + W$  is a solution to AX = B. Indeed,

$$A(\mathbf{v}_0 + \mathbf{w}) = A\mathbf{v}_0 + A\mathbf{w} = B + 0 = B.$$

Moreover, if  $A\mathbf{v} = B$  then put  $w = \mathbf{v} - \mathbf{v}_0$ . We see that  $\mathbf{v} = \mathbf{v}_0 + \mathbf{w}$  and

